NOTE ON MAXIMA AND MINIMA BY ALGEBRAIC METHODS.

By N. J. LENNES, Massachusetts Institute of Technology.

The pedagogical interest of this note seems sufficient to warrant its publication. The problems on maxima and minima in the Calculus are admittedly of considerable value in that they furnish interesting and concrete applications of the theory of derivatives, and also because of the importance of the problems themselves. Since nearly all problems of this class, which involve algebraic functions only and which are readily soluable by finding values which make the first derivatives vanish, may also be solved very easily by elementary algebraic methods, it would seem desirable to do so. And all the more so since the algebraic method affords an excellent illustration of the use of the graph and besides brings into play the relations between the roots and the coefficients of an algebraic equation. To those who never reach the calculus but who do study college algebra it may be worth while, for the sake of these problems themselves, to solve them rather than an equal number of abstract exercises.

If the expression whose maximum or minimum we seek is of the second degree we need only to find the highest or lowest point of a curve

$$y=a_0x^2+a_1x+a_2$$

which may be done by solving this equation simultaneously with

$$y=b$$

and then giving b such value that the two roots are identical. That is, the line is made tangent to the curve.

If in a similar manner we seek the intersection points of the cubic

$$y=a_0x^3+a_1x^2+a_2x+a_3$$

and the line

$$y=b,$$

we have an equation yielding three roots. If the line is to be tangent to the curve at one point two of these three roots are coincident. Representing the coincident roots by r_1 and the remaining root by r_2 we have

$$\begin{cases} 2r_1+r_2=\dfrac{a_1}{a_0} \\ 2r_1r_2+r_1^2=\dfrac{a_2}{a_0} \end{cases}.$$

Of the two values of r_1 obtained by solving this set of equations, one clearly corresponds to a maximum and the other to a minimum. Indeed, this is at once evident to the eye by constructing a graph of the cubic.

In case a_0 is positive the maximum of the function $a_0x^3+a_1x^2+a_2x+a_3$ is to the *left* of its minimum, and hence the smaller value of r_1 obtained above corresponds to a maximum and the greater value to a minimum.

If $r_1=r_2$ or if these are both complex there is, of course, neither maximum nor minimum.

Clearly, the method applies to any curve whose equation is reducible to the form

$$x^3+a_1x^2+a_2x+f(y)=0.$$

where a_1 and a_2 are constants and $f(y)$ any real function whatever of y. It may, of course, be impossible to find the value of y at the maximum or minimum point, though the values of x for such point may always be found.

In a similar manner the maxima and minima points of the curve

$$x^n+a_1x^{n-1}+\ldots+a_{n-1}x+f(y)=0$$

may be found by solving an equation of the $(n-1)$st degree. In the case of the biquadratic the details are as follows:

Suppose two of the roots of the equation

$$x^4+a_1x^3+a_2x^2+a_3x+f(y)=b$$

are coincident. Denote the coincident roots by r_1 and the other two roots by r_2 and r_3. Then

$$\begin{cases} 2r_1+r_2+r_3=-a_1 & (1)\\ r_1^2+2r_1r_2+2r_1r_3+r_2r_3=a_2 & (2)\\ r_1^2r_2+r_1^2r_3+2r_1r_2r_3=-a_3 & (3)\end{cases}$$

From (1) and (2),

$$r_1^2+2r_1(-a_1-2r_1)+r_2r_3=a_2 \qquad (4)$$

From (1), (4) and (3),

$$r_1^2(-a_1-2r_1)+2r_1[a_2-r_1^2+2r_1(a_1+2r_1)]$$
$$=-a_3 \qquad (5)$$

or

$$4r_1^3+3a_1r_1^2+2a_2r_1=-a_3 \qquad (6)$$

The three values of r_1 obtained by solving equation (6) will be the values of x for which the given curve has turning points. The middle value will necessarily belong to a maximum and the other two to minima. In case two of the roots are coincident the remaining root will give a minimum and in case they are all coincident their value gives a minimum.

Again, these statements are at once rendered evident by means of the graph.